

# THE SINGULARITY CATEGORY OF A GENTLE ALGEBRA

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**ABSTRACT.** We determine the singularity category of an arbitrary finite dimensional gentle algebra  $\Lambda$ . It is a finite product of  $n$ -cluster categories of type  $\mathbb{A}_1$ . If  $\Lambda$  is a Jacobian algebra arising from a triangulation  $\Gamma$  of an unpunctured surface, then the number of factors equals the number of inner triangles of  $\Gamma$ . Moreover, our result yields a description of the singularity category of an  $\mathbb{A}_n$ -configuration of projective lines.

## 1. INTRODUCTION

Singularity categories were introduced and studied by Buchweitz [3]. Recently, Orlov's global version of singularity categories [8] attracted a lot of interest in geometry and theoretical physics. It is related to Kontsevich's Homological Mirror Symmetry Conjecture.

For Iwanaga–Gorenstein rings, Buchweitz gave an equivalent description in terms of stable categories of Gorenstein-projective modules (also known as maximal Cohen–Macaulay modules), see [3] and (2.2). In particular, the singularity category of a selfinjective algebra is just the stable module category, which was thoroughly studied in representation theory.

Gentle algebras are certain finite dimensional algebras (see Definition 2.1 below), whose (derived) module categories are well understood. For example, there is a complete classification of indecomposable objects in both categories. Examples include algebras which are derived equivalent to an  $\mathbb{A}_n$ -configuration of projective lines [4] and algebras coming from triangulations of an unpunctured surface [1], see Section 3 below. Moreover, the class of gentle algebras is closed under derived equivalence [10].

We give an explicit construction of all Gorenstein projective modules over gentle algebras. Using Buchweitz' equivalence, this yields a description of their singularity categories.

## 2. DEFINITIONS AND MAIN RESULT

Let  $k$  be a field. Let  $Q$  be a finite quiver with set of vertices  $Q_0$  and set of arrows  $Q_1$ . We read elements in the path algebra  $kQ$  from right to left. All modules are left modules.

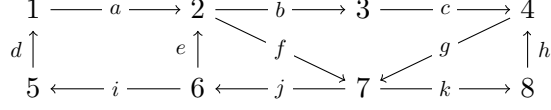
**Definition 2.1.** A *gentle algebra* is a finite dimensional algebra  $\Lambda = kQ/I$  such that:

- (G1) At any vertex, there are at most two incoming and at most two outgoing arrows.
- (G2) The admissible two-sided ideal  $I$  is generated by paths of length two.
- (G3) For each arrow  $\beta \in Q_1$ , there is at most one arrow  $\alpha \in Q_1$  such that  $\alpha\beta \in I$  and at most one arrow  $\gamma \in Q_1$  such that  $\beta\gamma \in I$ .
- (G4) For each arrow  $\beta \in Q_1$ , there is at most one arrow  $\alpha \in Q_1$  such that  $\alpha\beta \notin I$  and at most one arrow  $\gamma \in Q_1$  such that  $\beta\gamma \notin I$ .

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**Example 2.2.** An example of a gentle algebra  $\Lambda = kQ/I$  is given by the quiver  $Q$



with two-sided ideal  $I$  generated by the relations  $ba, fe, jf, ej, kg, hk$  and  $gh$ .

Geiß & Reiten [6] have shown that gentle algebras are *Iwanaga–Gorenstein rings*, i.e. the ring is Noetherian and has finite injective dimension as a left and as a right module over itself. For any Iwanaga–Gorenstein ring  $R$ , Zaks [12] has shown that  $\text{inj. dim}_R R = d = \text{inj. dim } R_R$  holds. We call  $d$  the *virtual dimension* of  $R$ . We are interested in the full subcategory of *Gorenstein projective*  $R$ -modules

$$\text{GP}(R) = \{M \in R\text{-mod} \mid \text{Ext}_R^i(M, R) = 0 \text{ for all } i > 0\}. \quad (2.1)$$

Let us list some well-known facts about Gorenstein-projective  $R$ -modules over Iwanaga–Gorenstein rings. Let throughout  $M$  and  $N$  be finitely generated  $R$ -modules.

- (GP1) A GP  $R$ -module is either projective or of infinite projective dimension.
- (GP2) If a GP  $R$ -module  $M$  contains a projective submodule  $P$ , then  $M \cong P \oplus M'$ .
- (GP3)  $M$  is GP if and only if  $M \cong \Omega^d(N)$  for some  $N$ . In particular, every Gorenstein projective module is a submodule of a projective module.
- (GP4)  $\text{GP}(R)$  is a Frobenius category with  $\text{proj GP}(R) = \text{proj } R$ .

Moreover, the embedding  $\text{GP}(\Lambda) \subseteq \mathcal{D}^b(\text{mod } -\Lambda)$  induces a triangle equivalence (see [3])

$$\underline{\text{GP}}(R) \longrightarrow \mathcal{D}_{sg}(R) := \frac{\mathcal{D}^b(\text{mod } -R)}{K^b(\text{proj } -R)}, \quad (2.2)$$

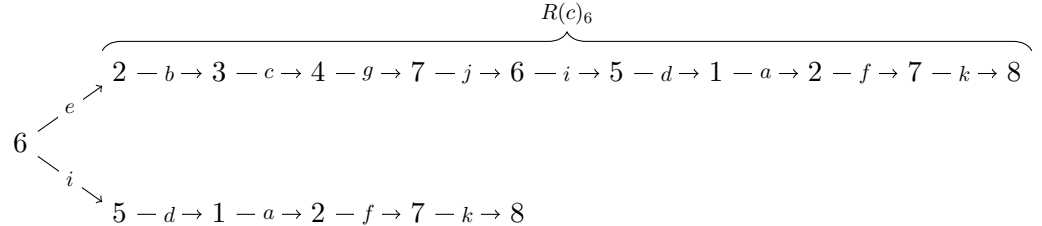
where the triangulated quotient category  $\mathcal{D}_{sg}(R)$  is called the *singularity category* of  $R$ .

For a gentle algebra  $\Lambda = kQ/I$ , we denote by  $\mathcal{C}(\Lambda)$  the set of repetition free cyclic path  $\alpha_1 \dots \alpha_n$  in  $Q$  such that  $\alpha_i \alpha_{i+1} \in I$  for all  $i$ , where we set  $n+1=1$ . In Example 2.2, we have  $\mathcal{C}(\Lambda) = \{jfe, kgh\}$ . Let  $i \in Q_0$  be a vertex lying on a cycle  $c \in \mathcal{C}(\Lambda)$  and  $P_i = \Lambda e_i$  be the corresponding indecomposable projective  $\Lambda$ -module. The radical of  $P_i$  has at most two direct summands  $R_1$  and  $R_2$ . We consider the radical embedding, where one of  $R_1$  and  $R_2$  may be zero and exactly one of the arrows  $\iota_1$  and  $\iota_2$  lies on the cycle  $c$

$$R_1 \oplus R_2 \xrightarrow{\begin{pmatrix} \cdot \iota_1 & \cdot \iota_2 \end{pmatrix}} P_i. \quad (2.3)$$

Let  $R(c)_i := R_j$  be the corresponding direct summand of the radical.

**Example 2.3.** In Example 2.2, we consider the vertex 6 lying on the cycle  $c = jfe$ . The indecomposable projective  $\Lambda$ -module  $P_6$  and its radical summand  $R(c)_6$  have the form



The following proposition is the main result of this note:

**Proposition 2.4.** *Let  $\Lambda = kQ/I$  be a finite dimensional gentle algebra.*

(a) *The indecomposable Gorenstein projective modules are given by*

$$\text{ind GP}(\Lambda) = \text{ind proj } \Lambda \cup \{R(c)_{s(\alpha_1)}, \dots, R(c)_{s(\alpha_n)} \mid c = \alpha_1 \dots \alpha_n \in \mathcal{C}(\Lambda)\}. \quad (2.4)$$

(b) *There is an equivalence of triangulated categories*

$$\mathcal{D}_{sg}(\Lambda) \cong \prod_{c \in \mathcal{C}(\Lambda)} \frac{\mathcal{D}^b(k - \text{mod})}{[l(c)]}, \quad (2.5)$$

where  $l(\alpha_1 \dots \alpha_n) = n$  and  $\mathcal{D}^b(k)/[l(c)]$  denotes the triangulated orbit category, [7]. This category is also known as the  $(l(c) - 1)$ -cluster category of type  $\mathbb{A}_1$ , [11].

We prove this result in Section 4 below.

### 3. APPLICATIONS AND EXAMPLES

The following geometric example was pointed out by Igor Burban.

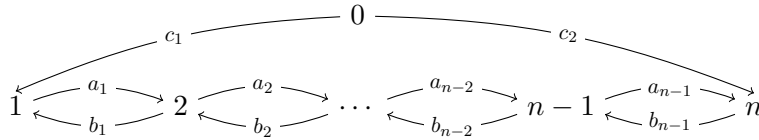
**Example 3.1.** Let  $\mathbb{X}_n$  be a chain of  $n$  projective lines



Using Buchweitz' equivalence (2.2) and Orlov's localization theorem [9], the singularity category of  $\mathbb{X}_n$  may be described as follows

$$(\mathcal{D}_{sg}(\mathbb{X}_n))^\omega := \left( \frac{\mathcal{D}^b(\text{Coh } \mathbb{X}_n)}{\text{Perf}(X)} \right)^\omega \cong \bigoplus_{i=1}^{n-1} \underline{\text{MCM}}(O_{nd}) \cong \bigoplus_{i=1}^{n-1} \frac{\mathcal{D}^b(k - \text{mod})}{[2]}, \quad (3.1)$$

where  $(-)^\omega$  denotes the idempotent completion [2] and  $\underline{\text{MCM}}(O_{nd})$  denotes the stable category of maximal Cohen–Macaulay modules over the nodal singularity  $O_{nd} = k[[x, y]]/(xy)$ . Burban [4] showed that  $\mathcal{D}^b(\text{Coh } \mathbb{X}_n)$  has a tilting bundle with endomorphism algebra  $\Lambda_n$



bounded by the relations  $a_i b_i = 0 = b_i a_i$  for all  $1 \leq i \leq n$ . Hence we have a triangle equivalence  $\mathcal{D}^b(\text{Coh } \mathbb{X}_n) \rightarrow \mathcal{D}^b(\text{mod} - \Lambda_n)$  inducing a triangle equivalence

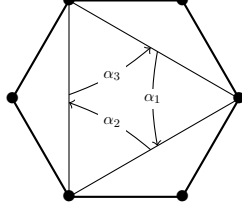
$$\mathcal{D}_{sg}(\mathbb{X}_n) \xrightarrow{\sim} \mathcal{D}_{sg}(\Lambda_n). \quad (3.2)$$

Since  $\Lambda_n$  is a gentle algebra, we can apply Proposition 2.4.  $\mathcal{C}(\Lambda_n)$  consists of  $n - 1$  cycles of length two. Therefore  $\mathcal{D}_{sg}(\Lambda)$  is equivalent to the right hand side of (3.1). In particular, we see that the singularity category  $\mathcal{D}_{sg}(\mathbb{X}_n)$  is idempotent complete.

Assem, Brüstle, Charbonneau-Jodoin & Plamondon [1] studied a class of gentle algebras  $A(S, \Gamma)$  arising from a triangulation  $\Gamma$  of a marked Riemann surface  $S = (S, M)$  without punctures. In particular, they show that the “inner triangles” of  $\Gamma$  are in bijection with the elements of  $\mathcal{C}(A(S, \Gamma))$ , which are all of length three. This has the following consequence.

**Corollary 3.2.** *In the notation above, the number of direct factors in the decomposition (2.5) of the singularity category  $\mathcal{D}_{sg}(A(S, \Gamma))$  equals the number of inner triangles of  $\Gamma$ .*

**Example 3.3.** A prototypical case is the hexagon  $S$  with six marked points on the boundary. We consider the following triangulation  $\Gamma$  with exactly one inner triangle.



The corresponding gentle algebra  $A(S, \Gamma)$  is a 3-cycle with relations  $\alpha_2\alpha_1 = 0$ ,  $\alpha_3\alpha_2 = 0$  and  $\alpha_1\alpha_3 = 0$ . This is a selfinjective algebra. Hence, the singularity category  $\mathcal{D}_{sg}(A(S, \Gamma))$  is triangle equivalent to the stable module category  $A(S, \Gamma) - \underline{\text{mod}}$ , by (2.2). This is in accordance with Proposition 2.4 and Corollary 3.2.

*Remark 3.4.* More generally, the algebras arising as Jacobian algebras from ideal triangulations of Riemann surfaces *with* punctures are usually of infinite global dimension. It would be interesting to study their singularity categories and relate them to the triangulation.

#### 4. PROOF

**4.1. Proof of part (a).** Let  $c \in \mathcal{C}(\Lambda)$  be a cycle, which we label as follows  $1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-1}} n \xrightarrow{\alpha_n} 1$ . Then there are short exact sequences

$$0 \rightarrow R(c)_i \rightarrow P_i \rightarrow R(c)_{i-1} \rightarrow 0, \quad (4.1)$$

for all  $i = 1, \dots, n$ . In particular, for every  $n \geq 0$ ,  $R(c)_i$  may be written as a  $n$ th-syzygy module  $\Omega^n(X)$ , for some  $\Lambda$ -module  $X$ . Thus,  $R(c)_i \in \text{GP}(\Lambda)$  by (GP3). Since projective modules are GP by definition, this shows the inclusion “ $\supseteq$ ” in (a).

It remains to show that there are no further Gorenstein-projective modules. For this, we use that the indecomposable modules over gentle algebras are classified. They are either string or band modules and are given by certain words in the alphabet  $\{\alpha, \alpha^{-1} \mid \alpha \in Q\}$ , see [5] for a detailed account. We claim that an indecomposable Gorenstein-projective  $\Lambda$ -module  $M$  containing a subword of the form  $\alpha\beta^{-1}$ , with  $\alpha \neq \beta$  is projective. We think of  $\alpha\beta^{-1}$  as a ‘roof’, where  $s, t, u$  are basis vectors of  $M$ , such that  $\alpha \cdot t = s$  and  $\beta \cdot t = u$ .

$$\begin{array}{ccc} & t & \\ \alpha \swarrow & & \searrow \beta \\ s & & u \end{array} \quad (4.2)$$

By (GP3),  $M$  is a submodule of some projective module  $P$ . Let  $U(t) \subset P$  be the submodule generated by the image of  $t$  in  $P$ . The properties (G1) and (G2) imply that  $U(t)$  is projective. Since  $M$  is indecomposable, (GP2) applied to  $U(t) \subseteq M$  shows that  $M \cong U(t)$  is projective. This shows the claim.

In particular, band modules cannot be GP (they always contain such a subword). By the shape of the projectives modules over a gentle algebra, submodules of projectives

cannot come from strings containing a subword of the form  $\alpha^{-1}\beta$ . Using (GP3) again, we have reduced the set of possible indecomposable GP  $\Lambda$ -modules to projective modules or direct strings  $S = \beta_n \dots \beta_1$ . We also allow  $S$  to consist of a single lazy path  $e_i$  (this corresponds to a simple module). Let  $M(S)$  be the corresponding  $\Lambda$ -module. If  $M(S)$  is properly contained in some projective module  $P$ , then there exists an arrow  $\alpha$  such that  $\beta_n \dots \beta_1 \alpha \notin I$  and  $\gamma \beta_n \dots \beta_1 \alpha \in I$  for every arrow  $\gamma \in Q_1$ . It follows that  $M(S)$  is a direct summand of the radical of  $P_{s(\alpha)}$ .

We have already said that the radical of an indecomposable projective  $\Lambda$ -module  $P$  has at most two indecomposable direct summands  $R_1$  and  $R_2$  (2.3). We need the following claim: if  $\iota_i$  does not lie on a cycle  $c \in \mathcal{C}(\Lambda)$  then  $R_i$  has finite projective dimension.

If  $R_i$  is not projective the situation locally looks as follows (we allow  $n$  to be zero)

$$\begin{array}{ccccccc} \dots & \xrightarrow{\iota_i} & \sigma & \xrightarrow{\beta_1} & \dots & \xrightarrow{\beta_n} & \bullet \\ & \searrow \psi_1 & \underbrace{\hspace{1.5cm}} & & & & \\ \dots & & & R_i & & & \end{array}$$

where  $\psi_1 \iota_i \in I$ . Moreover,  $\psi_1$  cannot lie on a cycle, since this would contradict our assumption on  $\iota_i$ . We have a short exact sequence

$$0 \rightarrow R' \xrightarrow{\psi_1} P_\sigma \rightarrow R_i \rightarrow 0 \quad (4.3)$$

where  $R'$  is a direct summand of the radical of  $P_\sigma$ .  $R'$  has the same properties as  $R_i$ , so we may repeat our argument. After finitely many steps, one of the occurring radical summands will be projective and the procedure stops. Indeed, otherwise we get a path  $\dots \psi_m \dots \psi_1 \iota_1$ , such that every subpath of length two is contained in  $I$ . Since there are only finitely many arrows in  $Q$ , this path is a cycle. Contradiction. Hence  $R_i$  has finite projective dimension. Thus it is GP if and only if it is projective, by (GP1).

We have shown that indecomposable GP modules are either projective or direct summands of the radical of some indecomposable projective, such that the radical embedding is defined by multiplication with an arrow on a cycle  $c \in \mathcal{C}(\Lambda)$ . This proves part (a).

**4.2. Proof of part (b).** By Buchweitz' equivalence (2.2), it suffices to describe the stable category  $\underline{\text{GP}}(\Lambda)$ . By part (a), the indecomposable objects in this category are precisely the radical summands  $R(c)_i$  for a cycle  $c \in \mathcal{C}$  and (4.1) shows that  $R(c)_i[1] \cong R(c)_{i-1}$ . In particular,  $R(c)_i[l(c)] \cong R(c)_i$ . It remains to prove that  $\underline{\text{Hom}}_\Lambda(R(c)_i, R(c')_j) = \delta_{ij} \delta_{cc'} \cdot k$ .  $R(c)_i$  is given by a string of the following form (it starts in  $\sigma$  and we allow  $n = 0$ )

$$\dots \xrightarrow{\iota_i} \sigma \xrightarrow{\beta_1} \dots \xrightarrow{\beta_n} \bullet, \quad (4.4)$$

Here,  $\iota_i$  is on the cycle  $c \in \mathcal{C}(\Lambda)$  and  $\beta_1 \iota_i \notin I$ , if  $n \neq 0$ . If there is a non-zero morphism of  $\Lambda$ -modules from  $R(c)_i$  to  $R(c')_j$ , then the latter has to be a string of the following form

$$R(c')_j: \quad \sigma' \xrightarrow{\beta'_1} \dots \xrightarrow{\beta'_m} \sigma \xrightarrow{\beta_1} \dots \xrightarrow{\beta_k} \bullet, \quad (4.5)$$

where we allow  $k = 0$  or  $m = 0$ . If both  $k$  and  $m$  are zero, then (G3) and our assumption that  $R(c')_j$  is a submodule of an indecomposable projective  $\Lambda$ -module imply that there is only one arrow starting in  $s$ . Namely, the arrow on the cycle. Hence,  $n = 0$  and therefore  $R(c)_i = R(c')_j$ . If  $k \neq 0$  and  $m = 0$ , then  $R(c)_i = R(c')_j$  as well.

In both cases,  $\text{End}_\Lambda(R(c)_i) \cong k$ . Note, that the simple module  $S_\sigma$  can appear (at most) twice as a composition factor of  $R(c)_i$ . However, in that case,  $R(c)_i$  locally has the following form  $\cdots \rightarrow \sigma \xrightarrow{\alpha} \cdots \rightarrow \bullet$ , where  $\alpha \neq \beta_1$  lies on the cycle  $c$  (see also Example 2.3). In particular, this does not yield additional endomorphisms.

If  $k \neq 0$  and  $m \neq 0$ , then it follows from (G4) that  $\beta'_m = \iota_i$ . If  $k = 0$ ,  $m \neq 0$  and  $\beta'_m \neq \iota_i$  then there are two different arrows ending in  $\sigma$ . Since  $\iota_i$  is on a cycle there is an arrow  $\gamma: \sigma \rightarrow \bullet$ , such that  $\gamma\iota_i \in I$ . It follows from (G3) that  $\gamma\beta'_m \notin I$ . Since  $R(c')_j$  is a submodule of a projective  $\Lambda$ -module the corresponding path starting in  $\sigma'$  has to be maximal. In particular, it does not end in  $\sigma$ . Contradiction. So we have  $\beta'_m = \iota_i$ .

In both cases our morphism factors over a projective module

$$R(c)_i \rightarrow P_{s(\iota_i)} \rightarrow R(c')_j \quad (4.6)$$

and therefore  $\underline{\text{Hom}}_\Lambda(R(c)_i, R(c')_j) = 0$ . This completes the proof.

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